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SUMS OF SQUARES IN REAL ANALYTIC RINGS

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ABSTRACT. Let A be an analytic ring. We show: (1) A has finite Pythagoras number if and only if its real dimension is ≤ 2, and (2) if every positive semidefinite element of A is a sum of squares, then A is real and has real dimension 2.

1. INTRODUCTION

In the study of positive semidefinite elements (= psd) and sums of squares (= sos) the two main problems are these:

- *Qualitative problem:* To know whether every positive semidefinite element is a sum of squares.
- *Quantitative problem:* To know whether there is $p \in \mathbb{N}$ such that every sum of squares is a sum of p squares.

These two problems have a meaning over any commutative ring A in terms of the real spectrum $\operatorname{Spec}_r(A)$: we consider the set $\mathcal{P}(A) \subset A$ of all $f \in A$ such that $f(\alpha) \geq 0$ for every prime cone $\alpha \in \operatorname{Spec}_r(A)$, and the set $\Sigma(A) \subset A$ of all sums of squares in A; the elements of $\mathcal{P}(A)$ are the psd's of A, and the elements of $\Sigma(A)$ are the sos's of A. Hence, $\Sigma(A) \subset \mathcal{P}(A)$ and the qualitative problem is whether $\mathcal{P}(A) = \Sigma(A)$. For the quantitative problem we have the *Pythagoras number* which is the smallest integer $p(A) = p \geq 1$ such that any sum of squares of A is a sum of p squares, and $p(A) = +\infty$ if such an integer does not exist. This is a very delicate invariant whose study has attracted a lot of attention from specialists in number theory, quadratic forms, real algebra and real geometry. The history of psd's and sos's is long and rich, and we refer the reader to [BCR] and [ChDLR] for further details. One particular case which is receiving more attention lately is that of local rings, see for instance [Sch2]. Here real algebra and the techniques of real spectra appear in essential ways. In this paper we deal with these matters for analytic rings.

An *analytic ring* (over $\mathbb{R}$) is a ring $A = \mathbb{R}\{x\}/I$, where I is an ideal of the ring $\mathbb{R}\{x\}$ of convergent power series in the indeterminates $x = (x_1, \dots, x_n)$. The ideal I defines a *zero set germ* $X = \mathcal{Z}(I) \subset \mathbb{R}^n$, and the elements of A can be seen as function germs on X. Note however that the ring of function germs on X is $\mathcal{O}(X) = \mathbb{R}\{x\}/\mathcal{J}(X)$, where $\mathcal{J}(X)$ stands for the *zero ideal of* X. The real Nullstellensatz says that $\mathcal{J}(X)$ is the *real radical* $\sqrt[r]{I}$ of I, hence $A = \mathcal{O}(X)$ if and

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only if A is *real*, and in general we only have a canonical epimorphism $A \to \mathcal{O}(X)$. In any case, by the real Positivstellensatz for germs the elements $f \in \mathcal{P}(A)$ are exactly the elements which are ≥ 0 on X. (For all of this see [AnBrRz, VIII.2,3]).

The most general known results are $p(A) = +\infty$ and $\mathcal{P}(A) \neq \Sigma(A)$ for any local regular ring A of dimension ≥ 3 ([ChDLR], [Sch2]); from this, it is not difficult to deduce that all real analytic rings of dimension ≥ 4 have the same properties (see [Rz3], [Fe1]). There are also a lot of results for curve germs ([Rz1], [CaRz] and [Or]) and for surface germs ([Rz3], [FeRz], [Fe1] and [Fe2]). But there is a serious lack of information without the regularity assumption for dimension 3. In this framework, our main results are the following. Let $A = \mathbb{R}\{x\}/I$ be an analytic ring, and $X = \mathcal{Z}(I)$ its zero set germ as above.

Theorem 1.1. *The Pythagoras number* $p(A)$ *is finite if and only if* $\dim X \leq 2$.

Theorem 1.2. *If* $\mathcal{P}(A) = \Sigma(A)$*, then* A *is real and* $\dim X \leq 2$.

We see now some aspects of their proofs.

Proof of the if part in 1.1. This is simple. We follow closely an idea in [ChDLR, 2.5].

Let $h_1, \dots, h_s$ be generators of $\mathcal{J}(X) = \sqrt[r]{I}$. There are $g_{ij} \in \mathbb{R}\{x\}$ such that

$$h_i^{2m_i} + g_{i1}^2 + \cdots + g_{ir}^2 \in I.$$

Consider the ideal $J = (h_1^{2m_1}, \dots, h_s^{2m_s}, I)$, so that $\sqrt{J} = \sqrt[r]{I}$, and let $B = \mathbb{R}\{x\}/J$. Then, $\dim B = n - \operatorname{ht}(J) = n - \operatorname{ht}(\sqrt{J}) = n - \operatorname{ht}(\sqrt[r]{I}) = \dim(X) \leq 2$ and B is a finitely generated $\mathbb{R}\{x_1, x_2\}$-module after a linear change of coordinates ([Rz2, II.2.3]). In [Fe2, 3.10] we see that the Pythagoras number of a ring which is generated as a module over $\mathbb{R}\{x_1, x_2\}$ by m elements is bounded by $2m$, hence $p(B) = p < +\infty$.

To conclude, let $a = a_1^2 + \ldots + a_q^2$ be a sum of squares in A. Then a is a sum of p squares in B and therefore

$$a = \alpha_1^2 + \cdots + \alpha_p^2 + \beta_1 h_1^{2m_1} + \cdots + \beta_s h_s^{2m_s} \quad \text{in } A.$$

Now, we pick real numbers $c_i > 0$ such that $c_i^2 + \beta_i(0) > 0$ and consequently $c_i^2 + \beta_i$ has a square root $\gamma_i \in A$. Therefore, since $h_i^{2m_i} + g_{i1}^2 + \cdots + g_{ir}^2 = 0$ in A we get

$$\beta_i h_i^{2m_i} = \gamma_i^2 h_i^{2m_i} + c_i^2 g_{i1}^2 + \cdots + c_i^2 g_{ir}^2.$$

Thus a is a sum of $p + s(r+1)$ squares in A, and so $p(A)$ is finite. □

Another elementary part of the above theorems is this:

Proof of the reality part in 1.2. This is a version of [Sch1, 6.3] and we include the argument because of its simplicity. Suppose that A is not real, that is, there exist $0 \neq h, g_1, \dots, g_\ell \in A$ such that $h^{2m} + g_1^2 + \cdots + g_\ell^2 = 0$. It is clear that $h, g_1, \dots, g_\ell \in \mathfrak{m}_A$; furthermore, $h|_X \equiv 0$ and then $h \in \mathcal{P}(A)$. We claim that if $\mathcal{P}(A) = \Sigma(A)$ then $h \in \bigcap_{k \in \mathbb{N}} \mathfrak{m}_A^k$, against the condition $h \neq 0$.

Indeed, since $h \in \mathcal{P}(A) = \Sigma(A)$ then $h = h_1^2 + \cdots + h_s^2$ in A. Thus $h_i \in \mathfrak{m}_A$ and so $h \in \mathfrak{m}_A^2$. Furthermore, since $h|_X \equiv 0$ then $h_i|X \equiv 0$ and $h_i \in \mathcal{P}(A)$. Again $h_i = h_{i1}^2 + \cdots + h_{ir_i}^2$ where $h_{ij} \in \mathfrak{m}_A$ and $h_{ij}|_X \equiv 0$, thus $h \in \mathfrak{m}_A^4$. Repeating this, we conclude that $h \in \bigcap_{k \in \mathbb{N}} \mathfrak{m}_A^k$. □

Concerning the only if part of 1.1, note that $p(A) < +\infty$ implies $p(\mathcal{O}(X)) < +\infty$, via the epimorphism $A \to \mathcal{O}(X)$. Thus we can assume that $A = \mathcal{O}(X)$ is real. Therefore for both theorems we only have to consider *real* analytic rings. More precisely, let X be an analytic set germ (at the origin of $\mathbb{R}^n$). We denote by $\mathcal{P}(X)$ the set of all psd's on X and by $\Sigma(X)$ the set of all sums of squares in $\mathcal{O}(X)$, and let $p[X]$ stand for the Pythagoras number $p(\mathcal{O}(X))$.

We have reduced Theorems 1.1, 1.2 to prove the following:

Theorem 1.3. *Let X be a real analytic germ of dimension ≥ 3. Then $\mathcal{P}(X) \neq \Sigma(X)$ and $p[X] = +\infty$.*

This is done in several steps:

(i) Firstly, we see that it is enough to solve the case when X is irreducible.
(ii) Secondly, we use local parametrization to replace X by a hypersurface. This requires keeping track of a universal denominator.
(iii) Finally, we blow-up points and lines to achieve a regular situation:

Lemma 1.4. *Let $f \in \mathbb{R}\{x_1, \dots, x_n\}$ change sign at the origin and have no multiple factors. Then there exists a finite sequence T of local blowings-up of points and lines such that, after a linear change of coordinates, the strict transform of f is*

$$\widetilde{f \circ T} = (x_1 + g(x_2, \dots, x_n))\, U$$

where $g \in \mathbb{R}\{x_2, \dots, x_n\}$ has order ≥ 2 and $U \in \mathbb{R}\{x_1, \dots, x_n\}$ is a unit.

To illustrate this desingularization idea, we prove here an easier result concerning meromorphic germs. We denote by $\mathcal{M}(X)$ the ring of meromorphic function germs in X, which is the total ring of fractions of $\mathcal{O}(X)$ (and is a field when X is irreducible). For instance, $\mathcal{M}(\mathbb{R}^n)$ is the field of fractions $\mathbb{R}(\{x\})$ of $\mathbb{R}\{x\}$. Finally, let $p(X)$ stand for $p(\mathcal{M}(X))$. We have

Theorem 1.5. *Let X be a real analytic germ of dimension $d \geq 3$. Then $p(X) \geq d + 1$.*

Proof. We have $\mathcal{M}(X) = \prod \mathcal{M}(Y)$ where each Y is an irreducible component of X. Thus $p(X)$ is the maximum of the $p(Y)$'s and it is enough to prove the result for X irreducible. In that case, by [AnRz] there exists a regular local ring $B \subset K = \mathcal{M}(X)$ which dominates $A = \mathcal{O}(X)$ and has residue field $\mathbb{R}$ (this is local uniformization: B can even be obtained by a finite sequence of quadratic transforms). Let $u = (u_1, \dots, u_d)$ be a regular system of parameters of B, and consider the commutative diagram:

$$\begin{array}{ccccccc} A & \to & B & \to & \widehat{B} & = & \mathbb{R}[[u]] \\ & \searrow & \downarrow & & \downarrow & & \\ & & K & \to & \widehat{K} & = & \mathbb{R}((u)) \end{array}$$

where all the arrows are inclusions, and the $u_1, \dots, u_d$ can be seen as indeterminates over $\mathbb{R}$. Let $f \in \mathbb{R}[u] \subset B$ be a homogeneous polynomial which is sum of $d+1$ squares but not of d squares in $\mathbb{R}(u) \subset K$ (these polynomials exist for $d \geq 3$, [BCR, 6.4.20]). Now, if f was a sum of d squares in $K \subset \mathbb{R}((u))$, we would find $a_1, \dots, a_d, b \in \mathbb{R}[[u]]$ with

$$b^2 f = a_1^2 + \cdots + a_d^2.$$

Then, comparing initial forms in the expression above f could be written as a sum of d squares in $\mathbb{R}(u)$, which is impossible. $\square$

From these theorems, we see that real analytic germs with $\mathcal{P} = \Sigma$ and/or finite Pythagoras number must have dimension 2. However, it seems a very difficult matter to find them all. So far, we know the embedded surface germs with those properties (a small list of multiplicity 2 germs whose Pythagoras number is 2, [Rz3] and [Fe1]) and a few aditional examples of arbitrary multiplicity and embedding dimension (the Veronese cones, also with Pythagoras number 2, [Fe3]).

On the other hand, our results can be generalized in a quite straightforward way for local noetherian rings with *real closed residue field*. Let us stress that this condition on the residue field is in fact the most serious obstruction to get fully satisfactory general statements.

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2. Local uniformization of a hypersurface

The purpose of this section is to prove Lemma 1.4. Before that we need some notation and terminology from desingularization (simplified to the most).

2.1. **Strict transforms.** We will use homomorphisms $\varphi : \mathbb{R}\{x\} \to \mathbb{R}\{x\}$ of the following three types:

(a) linear changes: $\varphi(x) = Ax$,

(b) quadratic transforms: $\varphi(x) = (x_1, x_1x_2, \dots, x_1x_n)$,

(c) modified quadratic transforms: $\varphi(x) = (x_1, x_1x_2, x_1^\lambda x_3, \dots, x_1^\lambda x_n)$, $\lambda > 1$.

In other words, we use sequences of local blowings-up of points and lines, which up to linear changes have those formulas (a modified quadratic transform is the blowing-up of a point followed by $(\lambda - 1)$ many blowings-up of the same line).

Then for any series $f \in \mathbb{R}\{x\}$ and any sequence $T = [\varphi_1, \dots, \varphi_r]$ (the φ_i's as above) one has the *strict transform* $\widetilde{f \circ T}$ *of* f *via* T, defined step by step:

(a) for linear changes: $\widetilde{f \circ \varphi} = f \circ \varphi$,

(b) for quadratic transforms and modified quadratic transforms

$$\widetilde{f \circ \varphi} := (f \circ \varphi)/x_1^\mu,$$

where μ is the greatest integer such that x_1^μ divides $f \circ \varphi$.

If $f \in \mathbb{R}\{x\}$ is a series and φ a quadratic transform, then $f \circ \varphi = x_1^{\omega(f)}(\widetilde{f \circ \varphi})$ and $\omega(\widetilde{f \circ \varphi}) \leq \omega(f)$. Also, we recall here that:

(i) if f has no multiple factors then neither has $\widetilde{f \circ T}$,

(ii) each irreducible factor of $\widetilde{f \circ T}$ lies over one of f.

Furthermore, we have

Lemma 2.1. *Let $T = [\varphi_1, \dots, \varphi_r]$. Then there exist finitely many analytic series $q_1, \dots, q_\ell \in \mathbb{R}\{x\}$ such that for every $f \in \mathbb{R}\{x\}$ we have $f \circ T = q^\nu(\widetilde{f \circ T})$, where $q^\nu = q_1^{\nu_1} \cdots q_\ell^{\nu_\ell}$ for suitable integers $\nu_1, \dots, \nu_\ell \geq 0$. Moreover, $\widetilde{f \circ T}$ is relatively prime with all q_i's.*

Proof. We proceed by induction on r. For $r = 1$:

(i) if φ_1 is a linear change, then $f \circ T = \widetilde{f \circ T}$,

(ii) if φ_1 is a local blowing-up then $f \circ T = x_1^{\nu_1}\widetilde{f \circ T}$, and then x_1 and $\widetilde{f \circ T}$ are relatively prime.

Suppose now $r > 1$, and let $T_1 = [\varphi_1, \ldots, \varphi_{r-1}]$. By induction hypothesis there exist finitely many series $p_1, \ldots, p_\ell \in \mathbb{R}\{x\}$ so that for every $f \in \mathbb{R}\{x\}$

$$f \circ T_1 = p_1^{\nu_1} \cdots p_\ell^{\nu_\ell}(\widetilde{f \circ T_1}), \quad \nu_1, \ldots, \nu_\ell \geq 0,$$

and $\widetilde{f \circ T}$ is relatively prime with all p_i. Therefore,

$$f \circ T = f \circ T_1 \circ \varphi_r = (p_1 \circ \varphi_r)^{\nu_1} \cdots (p_\ell \circ \varphi_r)^{\nu_\ell}(\widetilde{f \circ T_1} \circ \varphi_r).$$

We distinguish again two cases:

(i) if φ_r is a linear change, we take $q_i = p_i \circ \varphi_r$ for all i.

(ii) if φ_r is a local blowing-up, factoring out all x_1's we get

$$f \circ T = x_1^{\nu_{\ell+1}}(\widetilde{p_1 \circ \varphi_r})^{\nu_1} \cdots (\widetilde{p_\ell \circ \varphi_r})^{\nu_\ell}(\widetilde{f \circ T})$$

and we take $q_i = \widetilde{p_i \circ \varphi_r}$ for $i = 1, \ldots, \ell$ and $q_{\ell+1} = x_1$.

It is clear from *(ii)* above that $\widetilde{f \circ T}$ is relatively prime with q_i for $i = 1, \ldots, \ell$ and with $q_{\ell+1}$ by the definition of the strict transform. □

Now, we prove Lemma 1.4 in two steps:

Proposition 2.2. *Let $f \in \mathbb{R}\{x\}$ change sign at the origin. Then there exists a sequence $T = [\varphi_1, \ldots, \varphi_\ell]$ such that the initial form of $\widetilde{f \circ T}$ is indefinite.*

Proof. For any $f \in \mathbb{R}\{x\}$ we will denote the following:

$r(f)$, the minimum order of a substitution $f(\alpha(t))$ with $f(\alpha(t)) < 0$ for $t > 0$,

$s(f)$, the minimum order of a substitution $f(\beta(t))$ with $f(\beta(t)) > 0$ for $t > 0$.

Both integers exist by the curve selection lemma and both are $\geq \omega(f)$. Let $\operatorname{In}(f)$ be the initial form of f. We claim that:

a) $\operatorname{In}(f)$ is indefinite if and only $r(f) = s(f) = \omega(f)$;

b) $\operatorname{In}(f)$ is positive semidefinite if and only if $s(f) = \omega(f) < r(f)$;

c) $\operatorname{In}(f)$ is negative semidefinite if and only if $s(f) > \omega(f) = r(f)$.

Indeed, if γ is a half-branch centered at the origin in $\mathbb{R}^n$, we can write $\gamma(t) = t^p\overline{\gamma}(t)$ with $p \geq 1$ and $\overline{\gamma}(0) \neq 0$. Then:

$$f(\gamma(t)) = t^{pm} \operatorname{In}(f)(\overline{\gamma}(0)) + \cdots, \qquad m = \omega(f).$$

Thus, if $r(f) = s(f) = m$, there exist half-branches $\alpha = t\overline{\alpha}$, $\beta = t\overline{\beta}$ with $\operatorname{In}(f)(\overline{\alpha}(0))$, $\operatorname{In}(f)(\overline{\beta}(0))$ not equal zero such that $f(\alpha(t)) < 0$, $f(\beta(t)) > 0$ for $t > 0$ small. Hence, $\operatorname{In}(f)(\overline{\alpha}(0)) < 0$, $\operatorname{In}(f)(\overline{\beta}(0)) > 0$ and we conclude that $\operatorname{In}(f)$ is indefinite. On the other hand, if there exist $\overline{\alpha} \in \mathbb{R}^n$ (resp. $\overline{\beta} \in \mathbb{R}^n$) such that $\operatorname{In}(f)(\overline{\alpha}) < 0$, (resp. $\operatorname{In}(f)(\overline{\beta}) > 0$), we consider the half-branch $\alpha(t) = \overline{\alpha}t$, (resp. $\beta(t) = \overline{\beta}t$) to get $s(f) = \omega(f)$ (resp. $r(f) = \omega(f)$). From these the equivalences follow.

Now, let $\lambda(f)$ denote the maximum of $r(f)$ and $s(f)$. We prove the proposition by double induction on all pairs $(\omega(f), \lambda(f))$ for $f \in \mathbb{R}\{x\}$ (changing sign at the origin), ordered lexicographically. Firstly, if $\omega(f) = 1$, $\operatorname{In}(f)$ is indefinite and we are done. Let $(1, 1) < (a, b)$ and suppose the result true for g with $(\omega(g), \lambda(g)) < (a, b)$. We are to show it for f with $(\omega(f), \lambda(f)) = (a, b)$.

If $a = b$, $\operatorname{In}(f)$ is indefinite again; so we can suppose $a < b$ and $\operatorname{In}(f)$ semidefinite. Changing f by $-f$ (if needed) we can suppose also that $\operatorname{In}(f)$ is psd. Then $s(f) = \omega(f) = a < b = r(f)$.

Now we select $\alpha(t)$ such that $f(\alpha(t)) < 0$ for $t > 0$ and $\omega(f(\alpha(t))) = b$. After a linear change L_1,

$$\alpha(t) = (t^{m_1}, a_2 t^{m_2} + \cdots, \dots, a_n t^{m_n} + \cdots),$$

where $1 \le m_1 < \ldots < m_n$, and after another

$$L_2(x_1, \dots, x_n) = (x_1 + c_1 x_n, \dots, x_{n-1} + c_{n-1} x_n, x_n)$$

f is regular of order a with respect to x_n. It is clear that α is now

$$\alpha(t) = (t^{m_1}, b_2 t^{m_2} + \cdots, \dots, b_n t^{m_n} + \cdots),$$

Set $\varphi(x_1, \dots, x_n) = (x_1, x_1 x_2, \dots, x_1 x_n)$ and consider

$$\gamma(t) = (t^{m_1}, b_2 t^{m_2 - m_1} + \cdots, \dots, b_n t^{m_n - m_1} + \cdots).$$

Thus, $g = \widetilde{f \circ \varphi}$ has the following properties:

(i) $g = \dfrac{f \circ \varphi}{x_1^a}$ *and* $\omega(g) \le a$ by (2.1).

(ii) $\{g < 0\} \ne \emptyset$ *and* $r(g) < b$.

Indeed, since

$$g(\gamma(t)) = \frac{f \circ \varphi(\gamma(t))}{t^{a m_1}} = \frac{f \circ \alpha(t)}{t^{a m_1}} < 0 \quad \text{for } t > 0.$$

we conclude that $\{g < 0\} \ne \emptyset$ and that

$$r(g) \le \omega(g(\gamma(t))) = \omega(f(\alpha(t))) - a m_1 < \omega(f(\alpha(t))) = b.$$

(iii) $\{g > 0\} \ne \emptyset$ *and* $s(g) \le s(f)$. *Hence* $(\omega(g), \lambda(g)) < (a, b)$.

Indeed, since f is regular of order a with respect to x_n, there exist a Weierstrass polynomial P in x_n (of degree and order equal to a) and a unit U such that $f = PU$. Furthermore, $U(0) > 0$ because $\operatorname{In}(f)$ is psd. Then, if $f = f_m + f_{m+1} + \cdots$ we deduce

$$g = \sum_{k \ge m} f_k(1, x_2, \dots, x_n) x_1^{k-m} = (\widetilde{P \circ \varphi})(U \circ \varphi).$$

Therefore, since f_m is psd we have

$$\begin{aligned} 0 \le f_m(1, 0, \dots, 0, t) &= g(0, \dots, 0, t) \\ &= U(0)\,(\widetilde{P \circ \varphi})(0, \dots, 0, t) = U(0)\,(t^m + \sum_{j=1}^{m} a_j t^{m-j}) \ne 0 \end{aligned}$$

where $a_j \in \mathbb{R}$ and $U(0) > 0$. Whence, we conclude that $\{g > 0\} \ne \emptyset$ and that $s(g) \le \omega(g(0, \dots, 0, t)) \le m = \omega(f) = s(f) < r(f) = a$.

By induction hypothesis there exists a sequence T_1, such that $\widetilde{g \circ T_1}$ has indefinite initial form. Consequently, $T = [T_1, L_1, L_2, \varphi]$ is the sequence we sought. $\square$

Proposition 2.3. *(a) Let* $f_1 \in \mathbb{R}\{x, y\}$ *be a series in two variables without multiple factors such that* $\operatorname{In}(f_1)$ *is indefinite. Then there exists a sequence* $S = [\psi_1, \dots, \psi_s]$ *of linear changes and quadratic transforms such that* $\widetilde{f_1 \circ S}$ *is a series of order one.*

(b) Let $f \in \mathbb{R}\{x_1, \dots, x_n\}$ be a series without multiple factors such that $\mathrm{In}(f)$ *is indefinite. Then there exists a sequence $T = [\varphi_1, \dots, \varphi_s]$ of linear changes and quadratic transforms such that*

$$\widetilde{f \circ T} = (l(x_1, x_2) + g(x_2, \dots, x_n))\, U$$

where $l \neq 0$ is a linear form, $g \in \mathbb{R}\{x_2, \dots, x_n\}$ is an analytic series of order ≥ 2 and $U \in \mathbb{R}\{x_1, \dots, x_n\}$ is a unit.

Proof. (a) In our hypotheses there exists a linear change ψ_1 such that $f_1 \circ \psi_1 = Q_1 \cdots Q_r U$ where $U \in \mathbb{R}\{x, y\}$ is a unit and the $Q_i \in \mathbb{R}\{y\}[x]$ are irreducible Weierstrass polynomials with degree equal to order and pairwise different. Furthermore, since $\mathrm{In}(f_1)$ is indefinite we can also suppose that $\mathrm{In}(Q_1)$ is indefinite. Then $Q_1 = 0$ contains a real curve germ at the origin, and working with it as in [JP, 5.3.8], there exist a sequence $S_1 = [\psi_1, \dots, \psi_\ell]$ of linear changes and quadratic transforms such that $\widetilde{Q_1 \circ S_1}$ has order 1. Let $h = \widetilde{f_1 \circ S_1}$. If h has order 1 we are done, so suppose that $\omega(h) > 1$. After a linear change $\psi_{\ell+1}$ we can assume

$$h = (y - a(x))\, P(x, y)\, Q(x, y)\, U(x, y)$$

where $P, Q \in \mathbb{R}\{x\}[y]$ are Weierstrass polynomials, P irreducible and $U \in \mathbb{R}\{x, y\}$ a unit. Let α be a root of P (α is a Puiseux series with (possibly) complex coefficients); since h has no multiple factors (as had g), $\alpha \neq a$; say $a = \sum_{i \geq 1} a_i x^i$ and $\alpha = \sum_{k \geq 1} c_k x^{k/q}$.

We now work by induction on $\theta[\alpha - a] = \min\{k \in \mathbb{N} |\ \omega(\alpha - a) \leq k\}$. Suppose first $\theta[\alpha - a] = 1$, that is, $0 < \omega(\alpha - a) \leq 1$. Then $c_k \neq 0$ for some $k < q$ or $a_1 - c_q \neq 0$. Up to the linear change $\psi_{\ell+2}(x, y) = (x, y + a_1 x)$ we have

$$y - a = y - \sum_{i \geq 2} a_i x^i.$$

But P is the irreducible polynomial of α over $\mathbb{R}(\{x\})$, the quotient field of the ring $\mathbb{R}\{x\}$, and there are two cases:

(i) P has a root in a real closure of $\mathbb{R}(\{x\})$*. Then*

$$P = \prod_{j=1}^{q} \left(y - \sum_{k=1}^{q-1} c_k \eta^k x^{k/q} - (c_q - a_1)x - \sum_{k \geq q+1} c_k \eta^k x^{k/q} \right)$$

where $\eta = e^{2\pi i/q}$.

(ii) P does not have a root in a real closure of $\mathbb{R}(\{x\})$*. Then*

$$P = \prod_{j=1}^{q} \left(y - \sum_{k=1}^{q-1} c_k \eta^k x^{k/q} - (c_q - a_1)x - \sum_{k \geq q+1} c_k \eta^k x^{k/q} \right)$$
$$\cdot \prod_{j=1}^{q} \left(y - \sum_{k=1}^{q-1} \overline{c_k} \eta^k x^{k/q} - (\overline{c_q} - a_1)x - \sum_{k \geq q+1} \overline{c_k} \eta^k x^{k/q} \right)$$

where $\eta = e^{2\pi i/q}$.

Consider $\psi_{\ell+3}(x,y)=(x,xy)$. It is clear from the equations above that $\widetilde{P\circ\psi_{\ell+3}}$ is a unit. Therefore the order of

$$\widetilde{h\circ\psi_{\ell+3}}=(y-\sum_{i\geq 2}a_ix^{i-1})(\widetilde{P\circ\psi_{\ell+3}})(\widetilde{Q\circ\psi_{\ell+3}})(U\circ\psi_{\ell+3})$$

is smaller than $\omega(h)$ and $\widetilde{h\circ\psi_{\ell+3}}$ has an irreducible factor of order 1.

This completes the case $\theta=1$. Next, let $\theta[\alpha-a]=j>1$ or, equivalently, $j-1<\omega(\alpha-a)\leq j$. Then $\alpha=\sum_{i=1}^{j-1}a_ix^i+\sum_{k\geq q(j-1)+1}c_kx^{k/q}$. Let $\psi_{\ell+3}$ be as above. The irreducible polynomial of $\beta=\sum_{i=2}^{j-1}a_ix^{i-1}+\sum_{k\geq q(j-1)+1}c_kx^{k/q-1}$ over $\mathbb{R}(\{x\})$ is $\widetilde{P\circ\psi_{\ell+3}}$. Moreover $\theta[\beta-\sum_{i\geq 2}a_ix^{i-1}]\leq j-1$ and

$$\widetilde{h\circ\psi_{\ell+3}}=(y-\sum_{i\geq 2}a_ix^{i-1})(\widetilde{P\circ\psi_{\ell+3}})(\widetilde{Q\circ\psi_{\ell+3}})(U\circ\psi_{\ell+3})$$

so by induction hypothesis we are done.

(b) Suppose first that $f_1=f(x_1,x_2,0,\dots,0)$ is a series without multiple factors whose initial form is indefinite. Then by *(a)* there exists a finite sequence $S=[\psi_1,\dots,\psi_s]$ such that $\widetilde{f_1\circ S}$ has order 1. Let

(i) $\varphi_j(x_1,\dots,x_n)=(\psi_j(x_1,x_2),x_3,\dots,x_n)$, if ψ_j is a linear change,
(ii) $\varphi_j(x_1,\dots,x_n)=(x_1,x_1x_2,x_1^{\lambda}x_3,\dots,x_1^{\lambda}x_n)$, if $\psi_j(x_1,x_2)=(x_1,x_1x_2)$.

Let $T=[\varphi_1,\dots,\varphi_s]$. It follows easily that for λ large enough

$$\widetilde{f\circ T}=(l(x_1,x_2)+g(x_2,\dots,x_n))\,U$$

where l is a linear form, $g\in\mathbb{R}\{x_2,\dots,x_n\}$ has order ≥ 2 and $U\in\mathbb{R}\{x_1,\dots,x_n\}$ is a unit.

So we only have to get the condition on $f_1=f(x_1,x_2,0,\dots,0)$ by some linear change L. Firstly, there exist a linear change L_1 such that $f\circ L_1$ is regular with respect to x_1 of order $\omega(f)$ and then $f\circ L_1=PV$ where P is a Weierstrass polynomial with degree equal to order and V is a unit. Let

$$\Delta=\operatorname{Res}_{x_1}\left(P,\frac{\partial P}{\partial x_1}\right)$$

be the discriminant of P which is $\neq 0$ since f has no multiple factors. Since $\operatorname{In}(f)$ is indefinite, the same is true for $\operatorname{In}(P)$. Therefore, as $\Delta\neq 0$ there exist $a\in\mathbb{R}^n$ such that $\operatorname{In}(P)(a)<0$ and $\operatorname{In}(\Delta)(a)\neq 0$. Since $\Delta\in(x_2,\dots,x_n)\mathbb{R}\{x_2,\dots,x_n\}$, $a_i\neq 0$ for some $i\geq 2$; say $i=2$. We consider the linear change

$$L_2(x_1,\dots,x_n)=(x_1+a_1x_2,a_2x_2,x_3+a_3x_2,\dots,x_n+a_nx_2).$$

Let Δ' be the discriminant of $Q=P\circ L_2$. It is easy to see from the properties of the discriminant that $\Delta'=\Delta\circ L_2$. Furthermore,

$$\begin{aligned}\operatorname{Res}_{x_1}&\left(Q(x_1,x_2,0,\dots,0),\frac{\partial Q(x_1,x_2,0,\dots,0)}{\partial x_1}\right)=\Delta'(x_2,0,\dots,0)\\&=\Delta\circ L_2(x_2,0,\dots,0)=\Delta(a_2x_2,\dots,a_nx_2)=x_2^q\operatorname{In}(\Delta)(a)+\cdots\neq 0\end{aligned}$$

and therefore $f \circ L_1 \circ L_2(x_1, x_2, 0, \dots, 0)$ has no multiple factors and its initial form is indefinite, because

$$\begin{aligned}\mathrm{In}(f \circ L_1 \circ L_2)(x_1, x_2, 0, \dots, 0) &= V(0)\ \mathrm{In}(P \circ L_1 \circ L_2)(x_1, x_2, 0, \dots, 0)\\ &= V(0)\ \mathrm{In}(P(x_1 + a_1x_2, a_2x_2, \dots, a_nx_2))\end{aligned}$$

and

$$\begin{aligned}\mathrm{In}(P(x_1 + a_1x_2, a_2x_2, \dots, a_nx_2))(0,1) = \mathrm{In}(P)(a) < 0,\\ \mathrm{In}(P(x_1 + a_1x_2, a_2x_2, \dots, a_nx_2))(1,0) = 1 > 0.\end{aligned}$$

So $L = L_1 \circ L_2$ is the linear change we sought. □

Whence, Lemma 1.4 follows from 2.2 and 2.3 *(b)*.

3. Proof of the statements

The purpose of this section is to prove Theorem 1.3. The key result is the following.

Proposition 3.1. *Let $f, \Delta \in \mathbb{R}\{x\}$ be such that f has no multiple factors, changes sign at the origin and f, Δ are relatively prime. Then, there exist:*
a sequence of transforms $T = [\varphi_1, \dots, \varphi_r]$,
a substitution $\tau(x_2, \dots, x_n) = (h, x_2, \dots, x_n)$ with $h \in \mathbb{R}\{x_2, \dots, x_n\}$, $\omega(h) \geq 2$,
a linear change $\Gamma(x_2, \dots, x_n)$ in the variables $x_2, \dots, x_n$, and
a quadratic transform $\rho(x_2, \dots, x_n) = (x_2, x_2x_3, \dots, x_2x_n)$, such that given

$$\Delta^2 g = \alpha_1^2 + \dots + \alpha_p^2 + f\beta, \quad g, \alpha_1, \dots, \alpha_p, \beta \in \mathbb{R}\{x_1, \dots, x_n\},$$

then the strict transform of $\widetilde{g \circ T} \circ \tau$ via $[\Gamma, \rho]$ is a sum of p squares in $\mathbb{R}\{x_2, \dots, x_n\}$.

Proof. First, by 1.4, there exist a sequence $T = [\varphi_1, \dots, \varphi_r]$ and a series $h \in \mathbb{R}\{x_2, \dots, x_n\}$ of order ≥ 2 such that $\widetilde{f \circ T} = (x_1 - h)U$ where $U \in \mathbb{R}\{x_1, \dots, x_n\}$ is a unit. Now, in view of 2.1 there exist finitely many analytic series $q_1, \dots, q_\ell \in \mathbb{R}\{x_1, \dots, x_n\}$ such that for every $G \in \mathbb{R}\{x_1, \dots, x_n\}$

$$G \circ T = q^\nu \cdot \widetilde{G \circ T}, \quad \text{for } \nu = (\nu_1, \dots, \nu_\ell)$$

and $\widetilde{G \circ T}$, q_i are relatively prime for all i.

Therefore, $\widehat{q_i} = q_i \circ \tau \neq 0$ for all i. Furthermore, since f, Δ are relatively prime so are $\widetilde{\Delta \circ T}$, $\widetilde{f \circ T}$; in particular, $x_1 = h$ is not a root of $\widetilde{\Delta \circ T}$ and $\widehat{\Delta} = \widetilde{\Delta \circ T} \circ \tau \neq 0$. So there exist a linear change Γ such that the series $\widehat{q_1}, \dots, \widehat{q_\ell}, \widehat{\Delta}$ are all regular with respect to x_2. Thus $\widehat{q_i} \circ \Gamma \circ \rho = x_2^{\omega(\widehat{q_i})} V_i$ and $\widehat{\Delta} \circ \Gamma \circ \rho = x_2^{\omega(\widehat{\Delta})} W$ where $W, V_i \in \mathbb{R}\{x_2, \dots, x_n\}$ are units (this an easy consequence of the fact that if P is a Weierstrass polynomial with respect to x_2 of degree equal to order, then $(\widetilde{P \circ \rho})$ is a unit).

If we plug the sequence T into the equation

$$\Delta^2 g = \alpha_1^2 + \dots + \alpha_p^2 + f\beta,$$

we obtain

$$\begin{aligned} q^\nu \left(\widetilde{\Delta \circ T}\right)^2 (\widetilde{g \circ T}) = (\Delta^2 g) \circ T &= (\alpha_1')^2 + \dots + (\alpha_p')^2 + (\widetilde{f \circ T})\, \beta' \\ &= (\alpha_1')^2 + \dots + (\alpha_p')^2 + (x_1 - h(x_2, \dots, x_n))U\beta' \end{aligned}$$

with $\alpha'_1, \dots \alpha'_p, \beta' \in \mathbb{R}\{x\}$. Therefore, if we substitute $\tau : x_1 = h$ we have

$$\widehat{q}^{\,\nu}\,\widehat{\Delta}^2\,(\widetilde{g \circ T} \circ \tau) = (\alpha'_1 \circ \tau)^2 + \cdots + (\alpha'_p \circ \tau)^2.$$

Finally, if $M \in \mathbb{R}\{x_2, \dots, x_n\}$ denotes the strict transform of $\widetilde{g \circ T} \circ \tau$ via $[\Gamma, \rho]$ we have

$$x_2^{\mu}\, V^{\nu}\, W^2\, M = (\alpha'_1 \circ \tau \circ \Gamma \circ \rho)^2 + \cdots + (\alpha'_p \circ \tau \circ \Gamma \circ \rho)^2$$

for an integer $\mu \geq 0$ and therefore, there exist $a_1, \dots, a_p \in \mathbb{R}\{x_2, \dots, x_n\}$ such that

$$x_2^{\mu} M = a_1^2 + \cdots + a_p^2.$$

It follows that x_2^{μ} can be simplified, and we are done. □

Now we are finally ready to prove 1.3.

Proof of Theorem 1.3. Let X be an analytic germ of dimension $d \geq 3$ and let X_1 be an irreducible component of X of the same dimension as X. By local parametrization ([Rz2, II.3.4]), after a linear change, there exist an irreducible Weierstrass polynomial $f \in \mathbb{R}\{x_1, \dots, x_d\}[x_{d+1}]$ with degree p equal to order, and discriminant $\Delta \in \mathbb{R}\{x_1, \dots, x_d\}$ such that the canonical homomorphism $A = \mathbb{R}\{x_1, \dots, x_d\} \to \mathcal{O}(X_1)$ is injective and finite and θ_{d+1}, the class of $x_{d+1} \mod \mathcal{J}(X_1)$, is a primitive element of the quotient field of $\mathcal{O}(X_1)$ over the quotient field of A. Then, by [Rz2, II.3.2],

$$\Delta \cdot \mathcal{O}(X_1) \subset A + A\theta_{d+1} + \cdots + A\theta_{d+1}^{p-1} \cong \mathcal{O}(X'_1),$$

where $X'_1 \subset \mathbb{R}^{d+1}$ is the hypersurface germ $f = 0$. Let T, τ, Γ, ρ be as in 3.1.

We prove first that $\Sigma(X) \neq \mathcal{P}(X)$. Let $M = x_2^6 + x_3^4 x_4^2 + x_3^2 x_4^4 - 3x_2^2 x_3^2 x_4^2$ be the Motzkin polynomial which is a psd form on $\mathbb{R}^{d+1}$ that is not a sum of squares of polynomials ([BCR, 6.3.6]). We can easily construct a polynomial $g \in \mathbb{R}[x_1, \dots, x_{d+1}]$ such that the strict transform of $\widetilde{g \circ T} \circ \tau$ via $[\Gamma, \rho]$ is M. In fact, we do it step by step:

(i) For ρ, we consider $M_1 = x_2^6 M(x_2, x_3/x_2, x_4/x_2)$ which satisfies $\widetilde{M_1 \circ \rho} = M$.
(ii) For Γ, we take $M_2 = M_1 \circ \Gamma^{-1}$; it is clear that $M_2 \circ \Gamma = M_1$.
(iii) For τ, we consider again $M_2 \in \mathbb{R}[x_2, \dots, x_d]$ because $M_2 \circ \tau = M_2$.
(iv) For each φ_j, $j = r, \dots, 1$, in the sequence T we take:
(a) $M_{r-j+3} = M_{r-j+2} \circ \varphi_j^{-1}$ if φ_j is a linear change, and
(b) $M_{r-j+3} = x_1^{\mu} M_{r-j+2}\left(x_1, \dfrac{x_2}{x_1^{\lambda}}, \dots, \dfrac{x_n}{x_1^{\lambda}}\right)$ if φ_j is a modified quadratic transform, that is, $\varphi_j(x) = (x_1, x_1 x_2, x_1^{\lambda} x_3, \dots, x_1^{\lambda} x_{d+1})$, and where μ is the smallest integer such $M_{r-j+3} \in \mathbb{R}[x_1, \dots, x_n]$.
In the end $g = M_{r+2}$.

Since all our mappings are birational, g is a psd polynomial on $\mathbb{R}^{d+1}$, hence on $\mathbb{R}^n = \mathbb{R}^{d+1} \times \mathbb{R}^{n-d-1}$. Thus $g \in \mathcal{P}(X)$, and we claim that g is not an sos in $\mathcal{O}(X)$. Indeed, otherwise it would be one in $\mathcal{O}(X_1)$ and since $\Delta \cdot \mathcal{O}(X_1) \subset \mathcal{O}(X'_1)$, $\Delta^2 \cdot g$ would be a sum of squares in $\mathcal{O}(X_1)$. By 3.1 the strict transform of $\widetilde{(g \circ T)} \circ \tau$ via $[\Gamma, \rho]$, which is M, would be a sum of squares in $\mathbb{R}\{x_2, \dots, x_d\}$. Looking at the initial forms, we would conclude that M is a sos of polynomials, contradiction. The claim is proved.

For $p[X] = +\infty$, it is enough to prove $p[X_1] = +\infty$. To that end, one proceeds as above for every $p \geq 1$, using instead of M an homogeneous polynomial $P \in$

$\mathbb{R}[x_2, x_3, x_4]$ which is sum of $p+1$ squares but not of p squares ([ChDLR, §4]), and the conclusion is clear. $\square$

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